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Advances in Fractional Milne-Type Inequalities for Convexity and Their Applications

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Abstract

Fractional calculus serves as a fundamental framework in mathematics, applied sciences, and the study of integral inequalities, offering versatile tools for modeling complex phenomena. This study focuses on establishing new Milne-type inequalities for various classes of functions within the framework of Atangana-Baleanu fractional integrals. The main contributions include fractional Milne-type inequalities for bounded functions, derived as special cases of the presented theorems and supported by illustrative examples. Furthermore, the work presents applications to the $q-$digamma function and provides refined error estimates associated with the generalized Milne-type quadrature formula.

Keywords: Convex functions; Atangana-Baleanu fractional integrals; bounded function; $q-$digamma functions; Young's inequality.

1 Introduction

Convexity plays an important role across numerous mathematical and applied disciplines, including economics, optimization, game theory, and the study of integral inequalities. For sufficiently smooth functions, convexity is typically characterized by the sign of the second derivative. Owing to its vast applicability, the concept has undergone significant generalizations and extensions over time. Its intrinsic connection with inequality theory has been particularly influential, facilitating major advances in areas such as differential equations, numerical analysis, and optimization theory. This synergy has inspired the development of stable and accurate analytical and computational techniques capable of handling increasingly complex systems. For detailed discussions on quadrature formulas and related results, the reader is referred to [4, 8]. The fundamental definition of convexity is given as follows;

Definition 1.1. [19] *If $\xi : \dot{I} \to \mathbb{R}$ is convex on $\dot{I}$ and let $\dot{I}$ be a convex set on $\mathbb{R}$ then,*

$$\xi\left(\gamma\aleph + (1-\gamma)\kappa\right) \leq \gamma\xi(\aleph) + (1-\gamma)\xi(\kappa),$$

for all $\aleph, \kappa \in \dot{I}$ and $\gamma \in [0,1]$.

Integral inequalities are powerful analytical tools with wide-ranging applications in areas such as spectral analysis, approximation theory, statistics, and distribution theory. They play a key role in providing precise bounds and estimates, which are essential for analyzing mathematical models and addressing problems across multiple disciplines. In particular, research on numerical integration and its associated error estimates holds an important place in the mathematical literature. Many studies focus on deriving inequalities that yield error bounds for various classes of functions, including bounded and Lipschitz functions. These bounds are instrumental in assessing the accuracy and stability of numerical integration schemes. In this context, Simpson-type inequalities have been formulated for different classes of convex functions [24].

Fractional calculus represents a major advancement over classical analysis, as it extends integral and differential operators to non-integer orders. This generalization provides a powerful framework for modeling systems with memory and hereditary effects. It is particularly valuable for describing dynamic processes and complex materials. Its ability to capture intricate behaviors has led to widespread applications across scientific and engineering disciplines, demonstrating both versatility and significant impact. You may find detailed examples and real-world uses, [26, 21].

The study of error estimates for differentiable, twice differentiable, and $n-$times differentiable functions remains a fundamental aspect of mathematical analysis. In this regard, trapezoid-type inequalities have been established for several categories of functions, such as differentiable convex functions [15], bounded functions [12, 13], Lipschitz functions [3], functions of bounded variation [14], twice differentiable convex functions [28] and Ostrowski-type inequalities based on multistep linear kernel [2] have further refined existing bounds. Fractional extensions of trapezoid-type inequalities have been addressed in [30], while midpoint-type inequalities have been developed for fractional integrals [18], differentiable convex functions [20], and twice differentiable convex functions [29]. More recently, a growing body of work [17, 27] has focused on deriving significant new inequalities, including those of Simpson's type which is defined as,

$$\left|\frac{1}{3}\left[\frac{\xi(\aleph)+\xi(\kappa)}{2} + 2\xi\left(\frac{\aleph+\kappa}{2}\right) - \frac{1}{\kappa-\aleph}\int_{\aleph}^{\kappa}\xi(z)\,dz\right]\right| \leq \frac{(\kappa-\aleph)^4}{2880}\left\|\xi^{(4)}\right\|_{\infty},$$

where $\xi : [\aleph, \kappa] \to \mathbb{R}$ is a four time differentiable function on $(\aleph, \kappa)$ and

$$\left\|\xi^{(4)}\right\|_\infty = \sup_{x\in(\aleph,\kappa)} \left|\xi^{(4)}(x)\right| < \infty.$$

Definition 1.2. *Suppose that $\xi : [\aleph, \kappa] \to \mathbb{R}$ is a four-times continuously differentiable mapping on $(\aleph, \kappa)$, and let $\|\xi^{(4)}\|_\infty = \sup_{x\in(\aleph,\kappa)} |\xi^{(4)}(x)\, dx| < \infty$. Then, one has the inequality [6] as followed,*

$$\left|\frac{1}{3}\left[2\xi(\aleph) - \xi\left(\frac{\aleph+\kappa}{2}\right) + 2\xi(\kappa)\right] - \frac{1}{\kappa-\aleph}\int_\aleph^\kappa \xi(z)\, dz\right| \leq \frac{7(\kappa-\aleph)^4}{23040}\left\|\xi^{(4)}\right\|_\infty.$$

Definition 1.3. [22] *Let $\xi \in \acute{L}[\aleph, \kappa]$, $\rho > 0$ then the left and right sides of Riemann-Liouville fractional integrals is defined as:*

$$\dot{I}^{\rho}_{\aleph+}\xi(\gamma) = \frac{1}{\Gamma(\rho)}\int_{\aleph}^{\acute{L}} \left(\acute{L}-\gamma\right)^{\rho-1}\xi(\gamma)\, d\gamma, \qquad \acute{L} > \aleph,$$

$$\dot{I}^{\rho}_{\kappa-}\xi(\gamma) = \frac{1}{\Gamma(\rho)}\int_{\acute{L}}^{\kappa} \left(\gamma-\acute{L}\right)^{\rho-1}\xi(\gamma)\, d\gamma, \qquad \acute{L} < \kappa,$$

where, $\Gamma(\rho)$ is gamma function,

$$\dot{I}^{\rho}_{\aleph+}\xi(\gamma) = \dot{I}^{\rho}_{\kappa-}\xi(\gamma) = \xi(\gamma).$$

Theorem 1.1. [7] *Let $\xi : [\aleph, \kappa] \to \mathbb{R}$ be a differentiable function on I^o, $\aleph, \kappa \in I^o$ with $\aleph < \kappa$, where $\xi' \in L[\aleph, \kappa]$. If $|\xi'|$ is convex function on $[\aleph, \kappa]$, then we get the following inequality holds,*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\xi(\aleph) - \xi\left(\frac{\aleph+\kappa}{2}\right) + 2\xi(\kappa)\right] - \frac{2^{\rho-1}\Gamma(\rho+1)}{(\kappa-\aleph)^\rho}\left[J^{\rho}_{\aleph+}\xi\left(\frac{\aleph+\kappa}{2}\right) + J^{\rho}_{\aleph-}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\right| \\ &\leq \frac{(\kappa-\aleph)}{12}\left(\frac{\rho+4}{\rho+1}\right)\left(|\xi'(\aleph)| + |\xi'(\kappa)|\right).\end{aligned}$$

Proposition 1.1. *If we take $\rho = 1$ in Theorem 1.1, we have*

$$\left|\frac{1}{3}\left[2\xi(\aleph) - \xi\left(\frac{\aleph+\kappa}{2}\right) + 2\xi(\kappa)\right] - \frac{1}{\kappa-\aleph}\int_\aleph^\kappa \xi(z)\, dz\right| \leq \frac{5(\kappa-\aleph)}{24}\left(|\xi'(\aleph)| + |\xi'(\kappa)|\right).$$

Theorem 1.2. [7] *Let $\xi : [\aleph, \kappa] \to \mathbb{R}$ be a differentiable function on I^o, $\aleph, \kappa \in I^o$ with $\aleph < \kappa$, where $\xi' \in L[\aleph, \kappa]$. If $|\xi'|$ is an L-Lipschitz function on $[\aleph, \kappa]$, then we get the following inequality holds,*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\xi(\aleph) - \xi\left(\frac{\aleph+\kappa}{2}\right) + 2\xi(\kappa)\right] - \frac{2^{\rho-1}\Gamma(\rho+1)}{(\kappa-\aleph)^\rho}\left[J^{\rho}_{\aleph+}\xi\left(\frac{\aleph+\kappa}{2}\right) + J^{\rho}_{\aleph-}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\right| \\ &\leq \frac{(\kappa-\aleph)^2}{24}\left(\frac{\rho+8}{\rho+2}\right)L.\end{aligned}$$

Proposition 1.2. *If we take $\rho = 1$ in Theorem 1.2, we have*

$$\left|\frac{1}{3}\left[2\xi(\aleph) - \xi\left(\frac{\aleph+\kappa}{2}\right) + 2\xi(\kappa)\right] - \frac{1}{\kappa-\aleph}\int_\aleph^\kappa \xi(z)\, dz\right| \leq \frac{(\kappa-\aleph)^2}{8}L.$$

Definition 1.4. [11] *Let $H^{'}(\aleph,\kappa)$ be the Sobolev space of order one defined as,*

$$H^{1}(\aleph,\kappa)=\left\{g\in L^{2}(\aleph,\kappa):g'\in L^{2}(\aleph,\kappa)\right\},$$

where

$$L^{2}(\aleph,\kappa)=\left\{g(z):\left(\int_{\aleph}^{\kappa}g^{2}(z)\,dz\right)^{\frac{1}{2}}<\infty\right\}.$$

Let $\xi\in H^{'}(\aleph,\kappa)$, $\aleph<\kappa$, $\rho\in[0,1]$, then the notion of left derivative in the sense of Caputo-Fabrizio is defined as,

$$\left({}_{\aleph}^{CFD}D^{\rho}\xi\right)(z)=\frac{\beta(\rho)}{1-\rho}\int_{\aleph}^{z}\xi^{'}(\gamma)\,e^{\frac{-\rho(z-\gamma)^{\rho}}{1-\rho}}\,d\gamma,$$

where $z>\rho$ and the associated integral operator is

$$\left({}_{\aleph}^{CF}I^{\rho}\xi\right)(z)=\frac{1-\rho}{\beta(\rho)}\xi(z)+\frac{\rho}{\beta(\rho)}\int_{\aleph}^{z}\xi(\gamma)\,d\gamma,$$

where $\beta(\rho)>0$ is the normalization function satisfying $\beta(0)=\beta(1)=1$. For $\rho=0,\rho=1$, the left derivative is defined as follows, respectively,

$$\left({}_{\aleph}^{CFD}D^{0}\xi\right)(z)=\xi'(z),$$
$$\left({}_{\aleph}^{CFD}D^{1}\xi\right)(z)=\xi(z)-\xi(\aleph).$$

For the right derivative operator,

$$\left({}_{\kappa}^{CFD}D^{\rho}\xi\right)(z)=\frac{\beta(\rho)}{1-\rho}\int_{z}^{\kappa}\xi^{'}(\gamma)\,e^{\frac{-\rho(z-\gamma)^{\rho}}{1-\rho}}\,d\gamma,$$

where $z<\kappa$ and the associated integral operator is

$$\left({}^{CF}I_{\kappa}^{\rho}\xi\right)(z)=\frac{1-\rho}{\beta(\rho)}\xi(z)+\frac{\rho}{\beta(\rho)}\int_{z}^{\kappa}\xi(\gamma)\,d\gamma,$$

where $\beta(\rho)>0$ is a normalization function that satisfies $\beta(0)=\beta(1)=1$.

Theorem 1.3. [25] *Let $\xi:[\aleph,\kappa]\to\mathbb{R}$ be a differentiable function on I^{o}, $\aleph,\kappa\in I^{o}$ with $\aleph<\kappa$, where $\xi'\in L[\aleph,\kappa]$. If ξ' is an L-Lipschitz function on $[\aleph,\kappa]$, then we have*

$$\left|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{\beta(\rho)}{\rho(\kappa-\aleph)}\left[\left(\left({}_{\aleph}^{CF}I^{\rho}\xi\right)(z)+\left({}^{CF}I_{\kappa}^{\rho}\xi\right)(z)\right)\right]+\frac{2(1-\rho)}{\beta(\rho)}\xi(z)\right|$$
$$\leq\frac{7(\kappa-\aleph)^{2}}{24}L.$$

Theorem 1.4. [25] *Let $\xi:[\aleph,\kappa]\to\mathbb{R}$ be a differentiable function on I^{o}, $\aleph,\kappa\in I^{o}$ with $\aleph<\kappa$, where $\xi'\in L[\aleph,\kappa]$. If there exits constants $-\infty<m<M<+\infty$ such that $m\leq\xi'(\gamma)\leq M$, for all $\gamma\in[\aleph,\kappa]$, then we have*

$$\left|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{\beta(\rho)}{\rho(\kappa-\aleph)}\left[\left(\left({}_{\aleph}^{CF}I^{\rho}\xi\right)(z)+\left({}^{CF}I_{\kappa}^{\rho}\xi\right)(z)\right)\right]+\frac{2(1-\rho)}{\beta(\rho)}\xi(z)\right|$$
$$\leq\frac{5(\kappa-\aleph)}{24}(M-m).$$

The concept of the Atangana-Baleanu fractional integral operator was first established by Atangana and Baleanu [5] and it is defined in the following form.

Definition 1.5. [1, 5] *The fractional integral operator with non-local kernel of function $\xi \in H^{'}(\aleph, \kappa)$ is defined as,*

$$ {}_{\kappa}^{AB}I_{\gamma}^{\rho}\left(\xi\left(\gamma\right)\right)=\frac{1-\rho}{\beta\left(\rho\right)}\xi\left(z\right)+\frac{\rho}{\beta\left(\rho\right)\Gamma\left(\rho\right)}\int_{x}^{\gamma}\xi\left(z\right)\left(\gamma-z\right)^{\rho-1}dz, $$

and right hand sides of AB-fractional operator,

$$ {}_{\gamma}^{AB}I_{\aleph}^{\rho}\left(\xi\left(\gamma\right)\right)=\frac{1-\rho}{\beta\left(\rho\right)}\xi\left(z\right)+\frac{\rho}{\beta\left(\rho\right)\Gamma\left(\rho\right)}\int_{\gamma}^{\aleph}\xi\left(z\right)\left(z-\gamma\right)^{\rho-1}dz. $$

Recent research has advanced the theory of integral inequalities through the incorporation of fractional operators and kernel-based techniques. Milne- and Hermite-Hadamard-type inequalities have been developed using multiplicative calculus [32], and new fractal-fractional parametric inequalities have been introduced [10]. Bullen-type inequalities have been derived via fractional integral operators [16]. Additionally, trapezoid-type inequalities involving proportional Caputo-hybrid operators [31] and fractional Milne-type inequalities with applications in applied mathematics [9] have been formulated. Collectively, these studies underscore the continuing evolution of convexity-based and fractional inequality analysis in contemporary mathematical research.

The main aim of this work is to establish a new identity for the Atangana-Baleanu operator. Using this identity, we derive a fractional variant of Milne-type inequalities for functions whose absolute values of derivatives exhibit convexity. The approach incorporates well-known inequalities such as the power mean inequality and Hölder's inequality, as well as results for bounded functions, to obtain supporting estimates. Finally, several applications are presented, including refinements for special means and error bounds for quadrature formulas.

2 Main results

We begin by formulating a Milne-type integral identity for the Atangana–Baleanu fractional integral, which serves as a foundational component of this study.

Lemma 2.1. *Let a differentiable function on $(\aleph, \kappa)$ from $\xi : [\aleph, \kappa] \to \mathbb{R}$, with $\aleph < \kappa$. If $\xi^{'} \in \acute{L}[\aleph, \kappa]$ with $0 < \rho \leq 1$, then the following equality holds,*

$$ \begin{aligned} &\frac{1}{3}\left[2\xi\left(\aleph\right)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi\left(\kappa\right)\right]-\frac{2\left(1-\rho\right)}{\delta\left(\rho\right)}\xi\left(z\right)\\ &-\frac{2^{\rho-1}\delta\left(\rho\right)\Gamma\left(\rho\right)}{\left(\kappa-\aleph\right)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\\ &=\frac{\left(\kappa-\aleph\right)}{2^{\rho+1}\delta\left(\rho\right)\Gamma\left(\rho\right)}\left[\int_{0}^{1}\left(\gamma^{\rho}-\frac{4}{3}\right)\xi'\left[\left(1-\gamma\right)\aleph+\gamma\left(\frac{\aleph+\kappa}{2}\right)\right]d\gamma\right.\\ &\left.+\int_{0}^{1}\left(\gamma^{\rho}+\frac{1}{3}\right)\xi'\left[\left(1-\gamma\right)\left(\frac{\aleph+\kappa}{2}\right)+\gamma\kappa\right]d\gamma\right]. \end{aligned} \tag{1} $$

Proof. Integration by parts, each integral can be evaluated as follows,

$$\begin{aligned}
\dot{I}_1 &= \int_0^1 \left(\gamma^\rho - \frac{4}{3}\right) \xi' \left[(1-\gamma)\aleph + \gamma\left(\frac{\aleph+\kappa}{2}\right)\right] d\gamma \\
&= \left(\gamma^\rho - \frac{4}{3}\right) \xi \left.\frac{(1-\gamma)\aleph + \gamma\left(\frac{\aleph+\kappa}{2}\right)}{\left(\frac{\kappa-\aleph}{2}\right)}\right|_0^1 - \frac{2\rho}{\kappa-\aleph}\int_0^1 \gamma^{\rho-1}\xi\left[(1-\gamma)\aleph + \gamma\left(\frac{\aleph+\kappa}{2}\right)\right] d\gamma \\
&= \frac{2}{3(\kappa-\aleph)}\left[-\xi\left(\frac{\aleph+\kappa}{2}\right) + 4\xi(\aleph)\right] - \frac{2\rho}{\kappa-\aleph}\int_0^1 \gamma^{\rho-1}\xi\left[(1-\gamma)\aleph + \gamma\left(\frac{\aleph+\kappa}{2}\right)\right] d\gamma \\
&= \frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left\{\frac{2}{3(\kappa-\aleph)}\left[-\xi\left(\frac{\aleph+\kappa}{2}\right) + 4\xi(\aleph)\right]\right\} \\
&\quad - \frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left\{\frac{2^{\rho+1}\rho}{(\kappa-\aleph)^{\rho+1}}\int_{\aleph}^{\frac{\aleph+\kappa}{2}} \xi(y)\left(\frac{\aleph+\kappa}{2} - y\right)^{\rho-1} dy\right\} \\
&= \frac{1}{2^\rho\delta(\rho)\Gamma(\rho)}\left[-\frac{1}{3}\xi\left(\frac{\aleph+\kappa}{2}\right) + \frac{4}{3}\xi(\aleph)\right] - \frac{\rho}{(\kappa-\aleph)^\rho\delta(\rho)\Gamma(\rho)}\int_{\aleph}^{\frac{\aleph+\kappa}{2}} \xi(y)\left(\frac{\aleph+\kappa}{2} - y\right)^{\rho-1} dy.
\end{aligned} \tag{2}$$

Simple calculation analogously,

$$\begin{aligned}
\dot{I}_2 &= \int_0^1 \left(\gamma^\rho + \frac{1}{3}\right) \xi' \left[(1-\gamma)\left(\frac{\aleph+\kappa}{2}\right) + \gamma\kappa\right] d\gamma \\
&= \left(\gamma^\rho + \frac{1}{3}\right) \frac{\xi\left.(1-\gamma)\left(\frac{\aleph+\kappa}{2}\right) + \gamma\kappa\right|_0^1}{\left(\frac{\kappa-\aleph}{2}\right)} - \frac{2\rho}{\kappa-\aleph}\int_0^1 \gamma^{\rho-1}\xi\left[(1-\gamma)\left(\frac{\aleph+\kappa}{2}\right) + \gamma\kappa\right] d\gamma \\
&= \frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left\{\frac{2}{3(\kappa-\aleph)}\left[4\xi(\kappa) - \xi\left(\frac{\aleph+\kappa}{2}\right)\right]\right\} \\
&\quad - \frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left\{\frac{2^{\rho+1}\rho}{(\kappa-\aleph)^{\rho+1}}\int_{\frac{\aleph+\kappa}{2}}^{\kappa} \xi(y)\left(y - \frac{\aleph+\kappa}{2}\right)^{\rho-1} dy\right\} \\
&= \frac{1}{2^\rho\delta(\rho)\Gamma(\rho)}\left[\frac{4}{3}\xi(\kappa) - \frac{1}{3}\xi\left(\frac{\aleph+\kappa}{2}\right)\right] - \frac{\rho}{(\kappa-\aleph)^\rho\delta(\rho)\Gamma(\rho)}\int_{\frac{\aleph+\kappa}{2}}^{\kappa} \xi(y)\left(y - \frac{\aleph+\kappa}{2}\right)^{\rho-1} dy.
\end{aligned} \tag{3}$$

By adding (2) and (3) and adding $\frac{2(1-\rho)}{\delta(\rho)}\xi(z)$, we get

$$\begin{aligned}
I &= \left(\dot{I}_1 + \dot{I}_2\right) + \frac{2(1-\rho)}{\delta(\rho)}\xi(z) \\
&= \frac{1}{2^\rho\delta(\rho)\Gamma(\rho)}\left[\frac{4}{3}\xi(\aleph) - \frac{2}{3}\xi\left(\frac{\aleph+\kappa}{2}\right) + \frac{4}{3}\xi(\kappa)\right]
\end{aligned}$$

$$-\frac{1}{2^{\rho}(\kappa-\aleph)^{\rho}}\left[\frac{\rho}{\delta(\rho)\Gamma(\rho)}\int_{\frac{\aleph+\kappa}{2}}^{\kappa}\xi(y)\left(y-\frac{\aleph+\kappa}{2}\right)^{\rho-1}dy\right.$$
$$\left.+\frac{\rho}{\delta(\rho)\Gamma(\rho)}\int_{\aleph}^{\frac{\aleph+\kappa}{2}}\xi(y)\left(\frac{\aleph+\kappa}{2}-y\right)^{\rho-1}dy\right]+\frac{2(1-\rho)}{\delta(\rho)}\xi(z)$$
$$=\frac{2}{2^{\rho}\delta(\rho)\Gamma(\rho)}\left[\frac{2}{3}\xi(\aleph)-\frac{1}{3}\xi\left(\frac{\aleph+\kappa}{2}\right)+\frac{2}{3}\xi(\kappa)\right]$$
$$-\frac{1}{(\kappa-\aleph)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}{}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z).$$

Hence, by multiplying the above result by $\frac{2^{\rho}\delta(\rho)\Gamma(\rho)}{2}$, we obtain our required identity (1). □

Remark 2.1. *In the above Lemma 2.1 put $\rho=1$, we get*

$$\left|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{1}{\kappa-\aleph}\int_{\aleph}^{\kappa}\xi(u)\,du\right|$$
$$=\frac{\kappa-\aleph}{4}\left[\int_{0}^{1}\left(\gamma-\frac{4}{3}\right)\xi'\left[(1-\gamma)\aleph+\gamma\left(\frac{\aleph+\kappa}{2}\right)\right]d\gamma\right.$$
$$\left.+\int_{0}^{1}\left(\gamma+\frac{1}{3}\right)\xi'\left[(1-\gamma)\left(\frac{\aleph+\kappa}{2}\right)+\gamma\kappa\right]d\gamma\right].$$

We now proceed to state the principal inequality associated with convexity.

Theorem 2.1. *Let a differentiable function on $(\aleph,\kappa)$ from $\xi:[\aleph,\kappa]\to\mathbb{R}$, with $\aleph<\kappa$. If $\xi'\in\acute{L}[\aleph,\kappa]$ with $0<\rho\leq 1$ and $|\xi'|$ is a convex on $[\aleph,\kappa]$, then the following inequality holds,*

$$\left|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z)\right.$$
$$\left.-\frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}{}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\right|$$
$$\leq\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\left(\frac{2}{3}-\frac{1}{1+\rho}+\frac{1}{2+\rho}\right)(|\xi'(\aleph)|+|\xi'(\kappa)|)\right.$$
$$\left.+\left(\frac{5}{6}+\frac{1}{1+\rho}-\frac{2}{2+\rho}\right)\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|\right].$$

Proof. From Lemma 2.1 and convexity, it follows that,

$$\left|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z)\right.$$
$$\left.-\frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}{}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\right|$$

$$\leq \frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\int_0^1\left|\gamma^\rho-\frac{4}{3}\right|\left|\xi'\left[(1-\gamma)\aleph+\gamma\left(\frac{\aleph+\kappa}{2}\right)\right]\right|d\gamma\right.$$
$$\left.+\int_0^1\left|\gamma^\rho+\frac{1}{3}\right|\left|\xi'\left[(1-\gamma)\left(\frac{\aleph+\kappa}{2}\right)+\gamma\kappa\right]\right|d\gamma\right]$$
$$\leq \frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\int_0^1\left|\gamma^\rho-\frac{4}{3}\right|\left((1-\gamma)\left|\xi'(\aleph)\right|+\gamma\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|\right)d\gamma\right.$$
$$\left.+\int_0^1\left|\gamma^\rho+\frac{1}{3}\right|\left((1-\gamma)\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|+\gamma\left|\xi'(\kappa)\right|\right)d\gamma\right]$$
$$=\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\left(\frac{2}{3}-\frac{1}{1+\rho}+\frac{1}{2+\rho}\right)\left(\left|\xi'(\aleph)\right|+\left|\xi'(\kappa)\right|\right)\right.$$
$$\left.+\left(\frac{5}{6}+\frac{1}{1+\rho}-\frac{2}{2+\rho}\right)\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|\right].$$

The proof is completed. □

Corollary 2.1. *In Theorem 2.1, we use the further convexity of $|\xi'|$, we have*

$$\left|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z)\right.$$
$$\left.-\frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^\rho}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\right|$$
$$\leq\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left(\frac{7+13\rho}{12+12\rho}\right)\left[\left|\xi'(\aleph)\right|+\left|\xi'(\kappa)\right|\right].$$

Remark 2.2. *Choosing $\rho=1$ in the Corollary 2.1, we have*

$$\left|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{1}{\kappa-\aleph}\int_\aleph^\kappa\xi(u)\,du\right|\leq\frac{5(\kappa-\aleph)}{24}\left[\left|\xi'(\aleph)\right|+\left|\xi'(\kappa)\right|\right].$$

The graph representation of the error bound is illustrated in Figure 1.

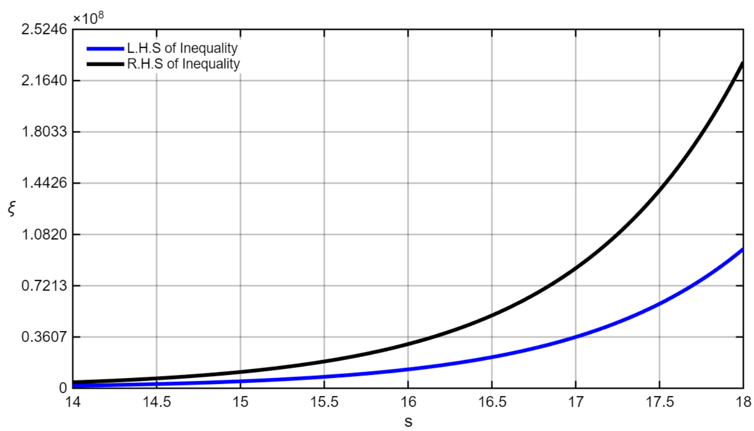

Figure 1: Graphical representation of the error bound in Remark 2.2, for $\xi(u)=e^u$ and $s\in[\aleph,\kappa]=[14,18]$, where s denotes the evaluation point used to compare the left and right hand sides of the inequality.

Theorem 2.2. *Let $\xi : [\aleph, \kappa] \to \mathbb{R}$ be a differentiable function on $(\aleph, \kappa)$ with $\aleph < \kappa$, $0 < \rho \leq 1$. If $\xi^{\,'} \in \acute{L}[\aleph, \kappa]$ and mapping $|\xi'|^q$ with $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, is a convex on $[\aleph, \kappa]$, then the following inequality holds,*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\xi(\aleph) - \xi\left(\frac{\aleph+\kappa}{2}\right) + 2\xi(\kappa)\right] - \frac{2(1-\rho)}{\delta(\rho)}\xi(z)\right. \\
&\left.- \frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right) + {}^{AB}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\right| \\
&\quad \leq \frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left(\frac{\rho\left(4^{p+1}-1\right)}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{\rho+1}\right)^{\frac{1}{q}} \\
&\qquad \times \left[\left(|\xi'(\aleph)|^q + \left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^q + |\xi'(\kappa)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. Again by using Lemma 2.1, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\xi(\aleph) - \xi\left(\frac{\aleph+\kappa}{2}\right) + 2\xi(\kappa)\right] - \frac{2(1-\rho)}{\delta(\rho)}\xi(z)\right. \\
&\left.- \frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right) + {}^{AB}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\right| \\
&\quad \leq \frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\int_0^1 \left|\gamma^{\rho} - \frac{4}{3}\right|\left|\xi'\left[(1-\gamma)\aleph + \gamma\left(\frac{\aleph+\kappa}{2}\right)\right]\right| d\gamma\right. \\
&\qquad \left. + \int_0^1 \left|\gamma^{\rho} + \frac{1}{3}\right|\left|\xi'\left[(1-\gamma)\left(\frac{\aleph+\kappa}{2}\right) + \gamma\kappa\right]\right| d\gamma\right].
\end{aligned} \tag{4}$$

Using Hölder's inequality in (4) and convexity of $|\xi'|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\xi(\aleph) - \xi\left(\frac{\aleph+\kappa}{2}\right) + 2\xi(\kappa)\right] - \frac{2(1-\rho)}{\delta(\rho)}\xi(z)\right. \\
&\left.- \frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right) + {}^{AB}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\right| \\
&\quad \leq \frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\left(\int_0^1 \left|\gamma^{\rho} - \frac{4}{3}\right|^p d\gamma\right)^{\frac{1}{p}}\left(\int_0^1 \left|\xi'\left[(1-\gamma)\aleph + \gamma\left(\frac{\aleph+\kappa}{2}\right)\right]\right|^q d\gamma\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \left(\int_0^1 \left|\gamma^{\rho} + \frac{1}{3}\right|^p d\gamma\right)^{\frac{1}{p}}\left(\int_0^1 \left|\xi'\left[(1-\gamma)\left(\frac{\aleph+\kappa}{2}\right) + \gamma\kappa\right]\right|^q d\gamma\right)^{\frac{1}{q}}\right] \\
&\quad \leq \frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\left(\int_0^1 \left|\gamma^{\rho} - \frac{4}{3}\right|^p d\gamma\right)^{\frac{1}{p}}\left(\int_0^1 \left((1-\gamma)|\xi'(\aleph)|^q + \gamma\left|\xi\left(\frac{\aleph+\kappa}{2}\right)\right|^q\right) d\gamma\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\int_0^1\left|\gamma^\rho+\frac{1}{3}\right|^p d\gamma\right)^{\frac{1}{p}}\left(\int_0^1\left((1-\gamma)\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^q+\gamma\left|\xi'(\kappa)\right|^q\right)d\gamma\right)^{\frac{1}{q}}\Bigg]$$

$$=\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left(\frac{\rho\left(4^{p+1}-1\right)}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{\rho+1}\right)^{\frac{1}{q}}$$

$$\times\left[\left(\left|\xi'(\aleph)\right|^q+\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^q+\left|\xi'(\kappa)\right|^q\right)^{\frac{1}{q}}\right].$$

The proof is completed. □

Remark 2.3. *In Theorem 2.2, by choosing $\rho=1$, then we have*

$$\left|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{1}{\kappa-\aleph}\int_\aleph^\kappa\xi(u)\,du\right|$$

$$\leq\frac{(\kappa-\aleph)}{4}\left(\frac{\left(4^{p+1}-1\right)}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\xi'(\aleph)\right|^q+\left|\xi'(\kappa)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\xi'(\aleph)\right|^q+3\left|\xi'(\kappa)\right|^q}{4}\right)^{\frac{1}{q}}\right].$$

The graph representation of the error bound is illustrated in Figure 2.

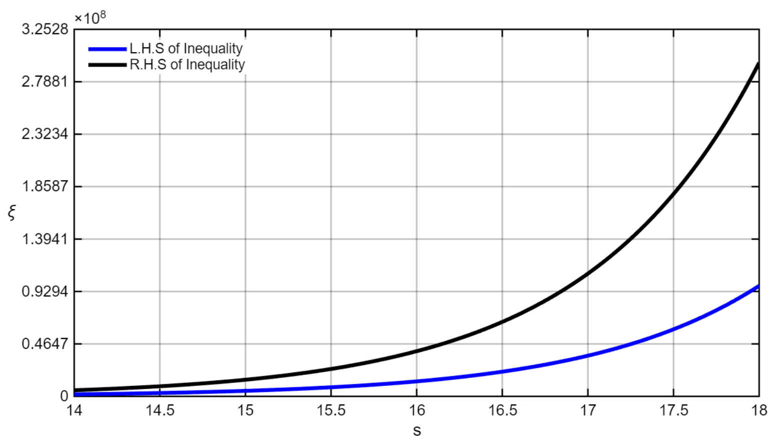

Figure 2: Graphical representation of the error bound in Remark 2.3, for $\xi(u)=e^u$ and $s\in[\aleph,\kappa]=[14,18]$, where s denotes the evaluation point used to compare the left and right hand sides of the inequality.

Theorem 2.3. *Let $\xi:[\aleph,\kappa]\to\mathbb{R}$ be a differentiable function on $(\aleph,\kappa)$ with $\aleph<\kappa$, $0<\rho\leq 1$. If $\xi'\in\acute{L}\,[\aleph,\kappa]$ and mapping $|\xi'|^q$ with $q\geq 1$ is a convex on $[\aleph,\kappa]$, then the following inequality holds,*

$$\Bigg|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z)$$

$$-\frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^\rho}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\Bigg|$$

$$\leq\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\left(\frac{4}{3}-\frac{1}{\rho+1}\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\left(\frac{2}{3}-\frac{1}{\rho+1}+\frac{1}{\rho+2}\right)\left|\xi'(\aleph)\right|^{q}+\left(\frac{2}{3}-\frac{1}{\rho+2}\right)\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{3}+\frac{1}{\rho+1}\right)^{1-\frac{1}{q}}\times\left(\left(\frac{1}{6}+\frac{1}{\rho+1}-\frac{1}{\rho+2}\right)\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^{q}+\left(\frac{1}{6}+\frac{1}{\rho+2}\right)\left|\xi'(\kappa)\right|^{q}\right)^{\frac{1}{q}}\Bigg].$$

Proof. Assuming $q\geq 1$ and using Lemma 2.1, along with the power mean integral inequality and convexity of $|\xi'|^{q}$, we derive

$$\Bigg|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z)$$
$$-\frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\Bigg|$$
$$\leq\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\int_0^1\left|\gamma^{\rho}-\frac{4}{3}\right|\left|\xi'\left[(1-\gamma)\aleph+\gamma\left(\frac{\aleph+\kappa}{2}\right)\right]\right|d\gamma\right.$$
$$\left.+\int_0^1\left|\gamma^{\rho}+\frac{1}{3}\right|\left|\xi'\left[(1-\gamma)\left(\frac{\aleph+\kappa}{2}\right)+\gamma\kappa\right]\right|d\gamma\right]$$
$$\leq\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\left(\int_0^1\left|\gamma^{\rho}-\frac{4}{3}\right|d\gamma\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\gamma^{\rho}-\frac{4}{3}\right|\left|\xi'\left[(1-\gamma)\aleph+\gamma\left(\frac{\aleph+\kappa}{2}\right)\right]\right|^{q}d\gamma\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|\gamma^{\rho}+\frac{1}{3}\right|d\gamma\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\gamma^{\rho}+\frac{1}{3}\right|\left|\xi'\left[(1-\gamma)\left(\frac{\aleph+\kappa}{2}\right)+\gamma\kappa\right]\right|^{q}d\gamma\right)^{\frac{1}{q}}\right]$$
$$\leq\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\left(\frac{4}{3}-\frac{1}{\rho+1}\right)^{1-\frac{1}{q}}\left(\left(\frac{2}{3}-\frac{1}{\rho+1}+\frac{1}{\rho+2}\right)\left|\xi'(\aleph)\right|^{q}+\left(\frac{2}{3}-\frac{1}{\rho+2}\right)\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{3}+\frac{1}{\rho+1}\right)^{1-\frac{1}{q}}\left(\left(\frac{1}{6}+\frac{1}{\rho+1}-\frac{1}{\rho+2}\right)\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^{q}+\left(\frac{1}{6}+\frac{1}{\rho+2}\right)\left|\xi'(\kappa)\right|^{q}\right)^{\frac{1}{q}}\right].$$

The proof is completed. ☐

Theorem 2.4. *Let $\xi:[\aleph,\kappa]\to\mathbb{R}$ be a differentiable function on $(\aleph,\kappa)$ with $\aleph<\kappa$, $0<\rho\leq 1$. If $\xi^{'}\in\acute{L}\,[\aleph,\kappa]$ and mapping $|\,\xi'|^{q}$ with $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, is a convex on $[\aleph,\kappa]$, then we get the following inequality holds,*

$$\Bigg|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z)$$
$$-\frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\Bigg|$$
$$\leq\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\frac{1}{p}\left(\frac{\rho\left(4^{p+1}-1\right)3^{-1-p}}{1+p}\right)\right.$$

$$+\frac{1}{q}\left(\left(\frac{|\xi'(\aleph)|^q+\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^q}{\rho+1}\right)+\left(\frac{\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^q+|\xi'(\kappa)|^q}{\rho+1}\right)\right)\Bigg].$$

Proof. From Lemma 2.1, we get

$$\begin{aligned}&\Bigg|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z)\\&-\frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^\rho}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}{}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\Bigg|\\&\le\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\int_0^1\left|\gamma^\rho-\frac{4}{3}\right|\left|\xi'\left[(1-\gamma)\aleph+\gamma\left(\frac{\aleph+\kappa}{2}\right)\right]\right|d\gamma\right.\\&\left.\quad+\int_0^1\left|\gamma^\rho+\frac{1}{3}\right|\left|\xi'\left[(1-\gamma)\left(\frac{\aleph+\kappa}{2}\right)+\gamma\kappa\right]\right|d\gamma\right].\end{aligned}$$

By using the Young's inequality, we have

$$\begin{aligned}&\Bigg|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z)\\&-\frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^\rho}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}{}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\Bigg|\\&\le\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\left(\frac{1}{p}\int_0^1\left|\gamma^\rho-\frac{4}{3}\right|^p d\gamma\right)+\frac{1}{q}\left(\int_0^1\left|\xi'\left((1-\gamma)\aleph+\gamma\left(\frac{\aleph+\kappa}{2}\right)\right)\right|d\gamma\right)^q\right.\\&\left.\quad+\left(\frac{1}{p}\int_0^1\left|\gamma^\rho+\frac{1}{3}\right|^p d\gamma\right)+\frac{1}{q}\left(\int_0^1\left|\xi'\left[(1-\gamma)\left(\frac{\aleph+\kappa}{2}\right)+\gamma\kappa\right]\right|d\gamma\right)^q\right]\\&\le\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\left(\frac{1}{p}\int_0^1\left|\gamma^\rho-\frac{4}{3}\right|^p d\gamma\right)+\frac{1}{q}\left(\int_0^1\left((1-\gamma)|\xi'(\aleph)|^q+\gamma\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^q\right)d\gamma\right)\right.\\&\left.\quad+\left(\frac{1}{p}\int_0^1\left|\gamma^\rho+\frac{1}{3}\right|^p d\gamma\right)+\frac{1}{q}\left(\int_0^1\left((1-\gamma)\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^q+\gamma|\xi'(\kappa)|^q\right)d\gamma\right)\right]\\&\le\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\frac{1}{p}\left(\frac{\rho\left(4^{p+1}-1\right)3^{-1-p}}{1+p}\right)\right.\\&\left.\quad+\frac{1}{q}\left(\left(\frac{|\xi'(\aleph)|^q+\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^q}{\rho+1}\right)+\left(\frac{\left|\xi'\left(\frac{\aleph+\kappa}{2}\right)\right|^q+|\xi'(\kappa)|^q}{\rho+1}\right)\right)\right].\end{aligned}$$

This completes the proof. □

Theorem 2.5. *Let $\xi:[\aleph,\kappa]\to\mathbb{R}$ be a differentiable function on $(\aleph,\kappa)$ with $\aleph<\kappa$, $0<\rho\le 1$ where $\xi'\in\acute{L}\,[\aleph,\kappa]$. If there exits constants $-\infty<m<M<+\infty$ such that $m\le\xi'(\gamma)\le M$, for all*

$\gamma \in [\aleph, \kappa]$, *then we have*

$$\left|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z)\right.$$
$$\left.-\frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}{}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\right|$$
$$\leq\frac{(\kappa-\aleph)(4\rho+1)}{2^{\rho+2}\delta(\rho)\Gamma(\rho)(\rho+2)}(M-m).$$

Proof. Using the Lemma 2.1, we have

$$\begin{aligned}&\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z)\\&-\frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}{}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\\&=\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\int_0^1\left(\gamma^{\rho}-\frac{4}{3}\right)\left(\xi'\left((1-\gamma)\aleph+\gamma\left(\frac{\aleph+\kappa}{2}\right)\right)\right)d\gamma\right.\\&\left.+\int_0^1\left(\gamma^{\rho}+\frac{1}{3}\right)\left(\xi'\left((1-\gamma)\left(\frac{\aleph+\kappa}{2}\right)+\gamma\kappa\right)\right)d\gamma\right]\\&=\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\left(\int_0^1\left(\gamma^{\rho}-\frac{4}{3}\right)\left(\xi'\left((1-\gamma)\aleph+\gamma\left(\frac{\aleph+\kappa}{2}\right)\right)\right)-\frac{m+M}{2}\right)d\gamma\right.\\&\left.+\left(\int_0^1\left(\gamma^{\rho}+\frac{1}{3}\right)\left(\xi'\left((1-\gamma)\left(\frac{\aleph+\kappa}{2}\right)+\gamma\kappa\right)\right)-\frac{m+M}{2}\right)d\gamma\right].\end{aligned}\tag{5}$$

By taking the mod of (5), we have

$$\begin{aligned}&\left|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z)\right.\\&\left.-\frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}{}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\right|\\&\leq\frac{(\kappa-\aleph)}{2^{\rho+1}\delta(\rho)\Gamma(\rho)}\left[\int_0^1\left|\gamma^{\rho}-\frac{4}{3}\right|\left|\xi'\left((1-\gamma)\aleph+\gamma\left(\frac{\aleph+\kappa}{2}\right)\right)-\frac{m+M}{2}\right|d\gamma\right.\\&\left.+\int_0^1\left|\gamma^{\rho}+\frac{1}{3}\right|\left|\left(\xi'\left((1-\gamma)\left(\frac{\aleph+\kappa}{2}\right)+\gamma\kappa\right)\right)-\frac{m+M}{2}\right|d\gamma\right].\end{aligned}$$

From $m \leq \xi'(\gamma) \leq M$ for $\gamma \in [\aleph, \kappa]$, we have

$$\left|\left(\xi'\left((1-\gamma)\aleph+\gamma\left(\frac{\aleph+\kappa}{2}\right)\right)\right)-\frac{m+M}{2}\right|\leq\frac{M-m}{2},\tag{6}$$

and

$$\left|\left(\xi'\left((1-\gamma)\left(\frac{\aleph+\kappa}{2}\right)+\gamma\kappa\right)\right)-\frac{m+M}{2}\right|\leq\frac{M-m}{2}.\tag{7}$$

By using (6) and (7), we have

$$\left|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{2(1-\rho)}{\delta(\rho)}\xi(z)\right.$$

$$-\frac{2^{\rho-1}\delta(\rho)\Gamma(\rho)}{(\kappa-\aleph)^{\rho}}\left[{}^{AB}_{\aleph}\dot{I}^{\rho}_{\frac{\aleph+\kappa}{2}}\xi\left(\frac{\aleph+\kappa}{2}\right)+{}^{AB}_{\frac{\aleph+\kappa}{2}}\dot{I}^{\rho}_{\kappa}\xi\left(\frac{\aleph+\kappa}{2}\right)\right]\Bigg|$$
$$\leq\frac{(\kappa-\aleph)}{2^{\rho+2}\delta(\rho)\Gamma(\rho)}\left(\int_0^1\left|\gamma^{\rho}-\frac{4}{3}\right|d\gamma+\int_0^1\left|\gamma^{\rho}+\frac{1}{3}\right|d\gamma\right)$$
$$\leq\frac{(\kappa-\aleph)(4\rho+1)}{2^{\rho+2}\delta(\rho)\Gamma(\rho)(\rho+2)}(M-m).$$

Hence proved. □

Remark 2.4. *If we choose $\rho=1$, in Theorem 2.5, we get*

$$\left|\frac{1}{3}\left[2\xi(\aleph)-\xi\left(\frac{\aleph+\kappa}{2}\right)+2\xi(\kappa)\right]-\frac{1}{\kappa-\aleph}\int_{\aleph}^{\kappa}\xi(u)\,du\right|\leq\frac{5(\kappa-\aleph)}{24}(M-m).$$

The graph representation of the error bound is illustrated in Figure 3.

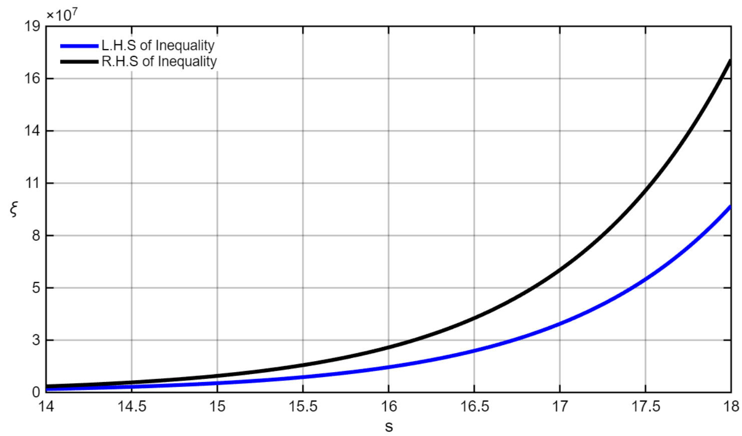

Figure 3: Graphical representation of the error bound in Remark 2.4, for $\xi(u)=e^{u}$ and $s\in[\aleph,\kappa]=[14,18]$, where s denotes the evaluation point used to compare the left and right hand sides of the inequality.

3 Applications

This section is devoted to applications involving the quadrature formula and $q-$digamma functions.

3.1 Application to quadrature formula

Let in the interval $[\sigma,\gamma]$ with partitioning $\mathbb{Z}$ is the set of points $\sigma=\acute{L}_0<\acute{L}_1<\acute{L}_2\ldots<\acute{L}_n=\gamma$, and consider

$$\int_{\sigma}^{\gamma}\nu(\nu)\,d\nu=\aleph(\nu,Y)+R(\nu,Y),$$

where

$$\aleph(\nu, Y) = \sum_{\dot{I}=0}^{n-1} \left(\frac{\nu_{\dot{I}+1} - \nu_{\dot{I}}}{3}\right) \left(2\nu(\nu_{\dot{I}}) - \nu\left(\frac{\nu_{\dot{I}} + \nu_{\dot{I}+1}}{2}\right) + 2\nu(\nu_{\dot{I}+1})\right),$$

and $R(\nu, Y)$ constitute the considering approximation error.

Proposition 3.1. *Let a differentiable function on (σ, γ) from $\nu : [\sigma, \gamma] \rightarrow \mathbb{R}$ with $0 \leq \sigma < \gamma$ and $\nu \in \acute{L}[\sigma, \gamma]$. If $\left|\nu'\right|$ is a convex, then the following inequality holds,*

$$R(\nu, Y) \leq \sum_{\dot{I}=0}^{n-1} \frac{(\nu_{\dot{I}+1} - \nu_{\dot{I}})}{2^{\rho+1}\delta(\rho)\Gamma(\rho)} \left(\frac{7 + 13\rho}{12 + 12\rho}\right) \left[\left|\nu'(\nu_{\dot{I}})\right| + \left|\nu'(\nu_{\dot{I}+1})\right|\right].$$

Proof. The result is obtained from the Theorem 4, applied to the sub intervals $[\nu_{\dot{I}} + \nu_{\dot{I}+1}]$ where $\left(\dot{I} = 0, 1, \ldots, n-1\right)$ of the partition of Z, with $\rho = \delta(0) = \delta(1) = 1$, we have

$$\begin{aligned} &\left|\frac{1}{3}\left(2\nu(\nu_{\dot{I}}) - \nu\left(\frac{\nu_{\dot{I}} + \nu_{\dot{I}+1}}{2}\right) + 2\nu(\nu_{\dot{I}+1})\right) - \frac{1}{\nu_{\dot{I}+1} - \nu_{\dot{I}}} \int_{\nu_{\dot{I}}}^{\nu_{\dot{I}+1}} \nu(u)\, du\right| \\ &\quad \leq \frac{(\nu_{\dot{I}+1} - \nu_{\dot{I}})}{2^{\rho+1}\delta(\rho)\Gamma(\rho)} \left(\frac{7 + 13\rho}{12 + 12\rho}\right) \left[\left|\nu'(\nu_{\dot{I}})\right| + \left|\nu'(\nu_{\dot{I}+1})\right|\right]. \end{aligned} \tag{8}$$

Multiplying both sides of (8) by $(\nu_{\dot{I}+1} - \nu_{\dot{I}})$, summing for $\dot{I} = 0, 1, \ldots, n-1$, and applying the triangular inequality, the desired result is obtained. $\square$

3.2 $q-$digamma functions

Let $0 < q < 1$, the $q-$digamma functions F, is the q analogue of the digamma function q defined as [23].

$$\begin{aligned} F &= -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+\nu}}{1 - q^{k+\nu}} \\ &= -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k\nu}}{1 - q^{k}}. \end{aligned}$$

The $q-$digamma functions F for $q > 1$ and $\nu > 0$, is defined as,

$$\begin{aligned} F &= -\ln(q-1) + \ln q \left[\nu - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-(k+\nu)}}{1 - q^{-(k+\nu)}}\right] \\ &= -\ln(q-1) + \ln q \left[\nu - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-k\nu}}{1 - q^{-k}}\right]. \end{aligned}$$

Proposition 3.2. *Let $\aleph, \kappa \in \mathbb{R}$ with $0 \leq \aleph < \kappa$, then we have*

$$\left|\frac{1}{3}\left[2\varphi_q'(\aleph) - \varphi_q'\left(\frac{\aleph + \kappa}{2}\right) + 2\varphi_q'(\kappa)\right] - \frac{1}{\kappa - \aleph}\int_{\aleph}^{\kappa} \varphi_q'(u)\, du\right| \leq \frac{5(\kappa - \aleph)}{24} \left[\left|\varphi_q''(\aleph)\right| + \left|\varphi_q''(\kappa)\right|\right].$$

Proof. The result is obtained from the Corollary 2.2, $\nu(\varepsilon) = \varphi_q^{'}(\varepsilon)$, $\varepsilon > 0$, $\nu(\varepsilon) = \varphi_q^{''}(\varepsilon)$ is convex. □

Proposition 3.3. *Let* $\aleph, \kappa \in \mathbb{R}$ *with* $0 \leq \aleph < \kappa$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$ *then we have*

$$\left|\frac{1}{3}\left[2\varphi_q^{'}(\aleph) - \varphi_q^{'}\left(\frac{\aleph+\kappa}{2}\right) + 2\varphi_q^{'}(\kappa)\right] - \frac{1}{\kappa-\aleph}\int_{\aleph}^{\kappa} \varphi_q^{'}(u)\,du\right|$$
$$\leq \frac{\kappa-\aleph}{4}\left(\frac{\left(4^{p+1}-1\right)}{3^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\varphi_q^{''}(\aleph)\right|^q + \left|\varphi_q^{''}(\kappa)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{\left|\varphi_q^{''}(\aleph)\right|^q + 3\left|\varphi_q^{''}(\kappa)\right|^q}{4}\right)^{\frac{1}{q}}\right].$$

Proof. The result is obtained from the Corollary 2.3, $\nu(\varepsilon) = \varphi_q^{'}(\varepsilon)$, $\varepsilon > 0$, $\nu(\varepsilon) = \varphi_q^{''}(\varepsilon)$. □

4 Conclusion

Fractional inequalities continues to be a vibrant research areas, offering valuable tools for modeling complex phenomena in natural and applied sciences. In this work, we introduce a new identity and formulate a fractional analogue of the Milne-type inequality within the framework of the Atangana-Baleanu fractional operator. Leveraging this identity, we derive refined bounds and estimates by employing classical inequalities such as the power mean inequality, Hölder's inequality, and results for bounded functions. Applications are also presented, including for the $q-$digamma function and error estimates in quadrature formulas. A graphical illustration is provided to visualize the analytical behavior of the established inequalities, highlighting the accuracy and consistency of the derived results.

Future research may focus on extending the present results by incorporating broader classes of convex functions, employing alternative fractional operators, or developing hybrid approaches that merge analytical and numerical techniques. Such directions could yield more generalized formulations, deeper theoretical insights, and wider applicability across mathematical analysis.

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References

[1] T. Abdeljawad & D. Baleanu (2016). Integration by parts and its applications of a new nonlocal fractional derivative with Mittag–Leffler nonsingular kernel. *Nonlinear Sciences and Ap-*

plications, *10*(3), 1098–1107. https://doi.org/10.22436/jnsa.010.03.20.

[2] U. Ali, M. D. Faiz, M. Muawwaz, S. Shabbir, A. Zaman & A. Qayyum (2025). A study of some new Ostrowski's type integral inequalities via multi-step linear kernel. *Journal of Inequalities and Mathematical Analysis*, *1*(2), 97–106. https://doi.org/10.63286/jima.010.

[3] M. W. Alomari (2013). A companion of the generalized trapezoid inequality and applications. *Journal of Mathematical Applications*, *36*, 5–15. https://doi.org/10.7862/rf.2013.1.

[4] M. W. Alomari & Z. Liu (2013). New error estimations for the Milne's quadrature formula in terms of at most first derivatives. *Konuralp journal of mathematics*, *1*(1), 17–23. https://izlik.org/JA97NH92DX.

[5] A. Atangana & D. Baleanu (2016). New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model. *Thermal Science*, *20*(2), 763–769. http://hdl.handle.net/11660/11244.

[6] A. D. Booth (1966). *Numerical Methods*. Butterworths Scientific Publications, California.

[7] H. Budak, P. Kosem & H. Kara (2023). On new Milne-type inequalities for fractional integrals. *Journal of Inequalities and Applications*, *2023*(1), Article ID: 10. https://doi.org/10.1186/s13660-023-02921-5.

[8] H. Budak & A.-A. Hyder (2023). Enhanced bounds for Riemann–Liouville fractional integrals: Novel variations of Milne inequalities. *AIMS Mathematics*, *8*(12), Article ID: 30760. https://doi.org/10.3934/math.20231572.

[9] H. Budak, H. Kara & H. Öğünmez (2025). On the fractional inequalities of the Milne type. *Mathematical Methods in the Applied Sciences*, *48*(11), 10833–10840. https://doi.org/10.1002/mma.10923.

[10] S. I. Butt & A. Khan (2023). New fractal–fractional parametric inequalities with applications. *Chaos, Solitons & Fractals*, *172*, Article ID: 113529. https://doi.org/10.1016/j.chaos.2023.113529.

[11] M. Caputo & M. Fabrizio (2015). A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation and Applications*, *1*(2), 73–85. https://www.naturalspublishing.com/files/published/0gb83k287mo759.pdf.

[12] P. Cerone & S. S. Dragomir (2019). Trapezoidal-type rules from an inequalities point of view. In *Handbook of Analytic Computational Methods in Applied Mathematics*, pp. 65–134. Chapman and Hall/CRC, Boca Raton, Florida.

[13] S. S. Dragomir (2000). On the midpoint quadrature formula for mappings with bounded variation and applications. *Kragujevac Journal of Mathematics*, *22*(1), 13–19.

[14] S. S. Dragomir (2001). On the trapezoid quadrature formula and applications. *Kragujevac Journal of Mathematics*, *23*(23), 25–36.

[15] S. S. Dragomir & R. P. Agarwal (1998). Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Applied Mathematics Letters*, *11*(5), 91–95. https://doi.org/10.1016/S0893-9659(98)00086-X.

[16] A. Fahad, S. I. Butt, B. Bayraktar, M. Anwar & Y. Wang (2023). Some new Bullen-type inequalities obtained via fractional integral operators. *Axioms*, *12*(7), Article ID: 691. https://doi.org/10.3390/axioms12070691.

[17] S. Hussain, J. Khalid & Y. M. Chu (2020). Some generalized fractional integral Simpson's type inequalities with applications. *AIMS Mathematics*, *5*(6), Article ID: 5859. https://www.aimspress.com/article/doi/10.3934/math.2020375.

[18] M. Iqbal, M. I. Bhatti & K. Nazeer (2015). Generalization of inequalities analogous to Hermite–Hadamard inequality via fractional integrals. *Bulletin of the Korean Mathematical Society*, *52*(3), 707–716. http://dx.doi.org/10.4134/BKMS.2015.52.3.707.

[19] J. L. W. V. Jensen (1906). Sur les fonctions convexes et les inégalités entre les valeurs moyennes. *Acta Mathematica*, *30*(1), 175–193. https://doi.org/10.1007/BF02418571.

[20] U. S. Kirmaci (2004). Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. *Applied Mathematics and Computation*, *147*(1), 137–146. https://doi.org/10.1016/S0096-3003(02)00657-4.

[21] B. Meftah, A. Lakhdari, W. Saleh & A. Kiliçman (2023). Some new fractal Milne-type integral inequalities via generalized convexity with applications. *Fractal and Fractional*, *7*(2), Article ID: 166. https://doi.org/10.3390/fractalfract7020166.

[22] K. S. Miller & B. Ross (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley & Sons, United States.

[23] D. S. Moak (1980). The $q-$gamma function for $q > 1$. *Aequationes Mathematicae*, *20*(1), 278–285. https://doi.org/10.1007/BF02190519.

[24] A. Munir, A. Qayyum, H. Budak, S. Qaisar, U. Ali & S. Supadi (2025). A fractional version of corrected dual-Simpson's type inequality via $s-$convex function with applications. *Malaysian Journal of Mathematical Sciences*, *19*(1), 17–33. https://doi.org/10.47836/mjms.19.1.02.

[25] A. Munir, A. Qayyum, L. Rathour, G. Atta, S. S. Supadi & U. Ali (2024). A study on Milne-type inequalities for a specific fractional integral operator with applications. *Korean Journal of Mathematics*, *32*(2), 297–314. https://doi.org/10.11568/kjm.2024.32.2.297.

[26] J. Nasir, S. Qaisar, S. I. Butt, K. A. Khan & R. M. Mabela (2022). Some simpson's Riemann–Liouville fractional integral inequalities with applications to special functions. *Journal of Function Spaces*, *2022*(1), Article ID: 2113742. https://doi.org/10.1155/2022/2113742.

[27] O. Rholam, M. Barmaki & D. Gretete (2023). Hermite–Hadamard inequalities type using fractional integrals for MT-convex stochastic process. *Malaysian Journal of Mathematical Sciences*, *17*(3), 473–485. https://doi.org/10.47836/mjms.17.3.14.

[28] M. Z. Sarikaya & N. Aktan (2011). On the generalization of some integral inequalities and their applications. *Mathematical and Computer Modelling*, *54*(9–10), 2175–2182. https://doi.org/10.1016/j.mcm.2011.05.026.

[29] M. Z. Sarikaya, A. Saglam & H. Yildirim (2010). New inequalities of Hermite–Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex. *International Journal of Open Problems in Computer Science and Mathematics*, *5*(3), 1–11. https://doi.org/10.12816/0006114.

[30] M. Z. Sarikaya, E. Set, H. Yaldiz & N. Başak (2013). Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Mathematical and Computer Modelling*, *57*(9–10), 2403–2407. https://doi.org/10.1016/j.mcm.2011.12.048.

[31] T. Tunç & I. Demir (2025). Some trapezoid-type inequalities for newly defined proportional Caputo-hybrid operator. *Journal of Inequalities and Mathematical Analysis*, *1*(1), 65–78. https://doi.org/10.63286/jima.2025.05.

[32] M. Umar, S. I. Butt & Y. Seol (2024). Milne and Hermite–Hadamard's type inequalities for strongly multiplicative convex function via multiplicative calculus. *AIMS Mathematics*, *9*(12), 34090–34108. https://doi.org/10.3934/math.20241625.